B2.1

Solutions to problem sheet 0

1. Let M be a non-zero submodule of V which is not equal to V. Let $v \in M$ be a nonzero vector and let $w \in V$ be a vector outside M. We claim that there is some $X \in M_n(\mathbb{C})$ such that $Xv = w$, so $w \in M$ which is a contradiction. To prove the claim let T_v, T_w be two invertible matrices with first column equal to v and w repectively. Let $e_1 \dots, e_n$ denote the standard basis of V we have $T_v(e_1) = v$ and $T_w(e_1) = w$. Therefore $T_w T_v^{-1}(v) = w$ and we can take $X = T_w T_v^{-1}$.

2. Suppose that $g^m = 1$. Then the minimal polynomial of g divides $x^m - 1$ and so has distinct roots (because $x^m - 1$ has no repeated roots). So g is diagonalizable by Part A linear algebra.

   If $char K = p$ this is no longer true: Let $g \in GL(2, K)$ be the upper triangular matrix with entries equal to 1 except at the lower left corner which is 0. Then $g^p = 1$ but g is not diagonalizable.

   Suppose that $n \leq p$ and the order of g is p^2. Let $a = g - 1$, i.e. $g = 1 + a$. We have $1 = g^{p^2} = (1+a)^{p^2} = 1 + a^{p^2}$ and hence $a^{p^2} = 0$ i.e. a is nilpotent matrix. The minimal polynomial of a must divide x^n and hence $a^n = 0$. Since $n \leq p$ we have $a^p = 1$ but then $g^p = (1+a)^p = 1 + a^p = 1$ so the order of g is p and not p^2, contradiction. So $n > p$.

3. Let V be the vector space over $\mathbb{R}$ (or any chosen field) with basis $B := \{b_g \mid g \in G\}$ labelled by the elements of G. For any $g \in G$ define a linear transformation $T_g : V \to V$ by its action on the basis B as follows:

$$T_g(v_x) = v_{gx} \quad \forall x \in G.$$

   (So T_g acts on B as a permutation in the same way as g acts on G by left multiplication). It is immediate that $T_{g_1 g_2} = T_{g_1} \circ T_{g_2}$ for all $g_1, g_2 \in G$ and so $\{T_g \mid g \in G\}$ is a subgroup of $GL(n, \mathbb{C})$ isomorphic to G (with $n = |G|$).

   Lastly let us show that A_5 is not isomorphic to a subgroup H of $GL(2, \mathbb{C})$. Suppose that $H \simeq A_5$ and consider an element $g \in H$ of order 2. By Q2 g must be diagonalizable with eigenvalues ± 1. If the eigenvalues are equal then g is $\pm Id$ and must commute with all elements of $H \simeq A_5$ which is not the case. Therefore g has eigenvalues 1 and -1 and in particular $det(g) = -1$. The determinant map restricted to H provides a homomorphism $\det : H \to (\mathbb{C}^*, \times)$ with a non-trivial image. Recall now that $H \simeq A_5$ is a simple group. Therefore the above homomorhism is injective and so H is isomorhic to its image $\det H \leq \mathbb{C}^*$. However the multiplicative group $\mathbb{C}^*$ is abelian and A_5 is not abelian, contradiction.